

On The Barometric Formulas And Their Derivation From Hydrodynamics and Thermodynamics

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Abstract

We derive the approximate pressure profiles, density profiles, and temperature profiles of an atmosphere, also called *barometric formulas*. Our variant of a derivation goes beyond the common standard exercise of a thermodynamics lecture, where commonly the discussion of the underlying physical assumptions is missed. We depart from the Navier-Stokes equation and explicitly point our attention on the physical assumptions disregarded elsewhere. We show that the usual assumptions can be relaxed leading to generalized formulas that hold even in the case of horizontal winds. This fundamental physics has some relevance to the current discussions on the climate debate.

Contents

1	Introduction	4
2	On the derivation of the barometric formulas	4
2.1	Input from hydrodynamics	4
2.2	Input from thermodynamics	7
2.3	The isothermal atmosphere	7
2.4	The adiabatic atmosphere	8
3	Results	11
	Appendix: Relevance to the current climate debate	12
	Acknowledgement	13
	References	13

1 Introduction

In the following, we derive approximate temperature profiles of an atmosphere, also called adiabatic lapse rates or better *barometric formulas*. Our variant of a derivation goes beyond the common standard exercise of a thermodynamics lecture, where commonly the discussion of the underlying physical assumptions is missed. We depart from the Navier-Stokes equation and explicitly point our attention on the physical assumptions disregarded elsewhere. By the way, this derivation is a good example on how to apply the magnetohydrodynamic equations regarded as redundant by some of our critics. Furthermore, it explicitly shows that in physics an application of formulas is valid *only in a finite space-time region*. In addition, we show that the usual assumptions can be relaxed leading to generalized formulas that hold even in the case of horizontal winds.

A brief historical review of the barometric formula is given in Ref. [1]. The reader is also referred to the textbook by Riegel and Bridger on “Fundamentals of Atmospheric Dynamics and Thermodynamics” [2].

2 On the derivation of the barometric formulas

2.1 Input from hydrodynamics

As described in our falsification paper [4, 3] the core of a climate model must be a set of equations describing the equations of fluid flow, namely the generalized Navier-Stokes equations. They describe the conservation of momentum and read

$$\frac{\partial}{\partial t}(\varrho \mathbf{v}) + \nabla \cdot (\varrho \mathbf{v} \otimes \mathbf{v}) = -\nabla p - \varrho \nabla \Phi + \varrho_e \mathbf{E} + \mathbf{j} \times \mathbf{B} + \nabla \cdot \mathbf{R} + \mathbf{F}_{\text{ext}} \quad (1)$$

where \mathbf{v} is the velocity vector field, p the pressure field, Φ the gravitational potential, \mathbf{R} the friction tensor, and \mathbf{F}_{ext} are the external force densities, which could describe the Coriolis and centrifugal accelerations. Neglecting the friction term and the electromagnetic fields we obtain the Euler equations.

Assumption 1

- We neglect the electromagnetic field terms.

We get the more common version of the Navier-Stokes equations

$$\frac{\partial}{\partial t}(\varrho \mathbf{v}) + \nabla \cdot (\varrho \mathbf{v} \otimes \mathbf{v}) = -\nabla p - \varrho \nabla \Phi + \nabla \cdot \mathbf{R} + \mathbf{F}_{\text{ext}} \quad (2)$$

The left hand side of this equation may be rewritten according to

$$\frac{\partial}{\partial t}(\varrho \mathbf{v}) + \nabla \cdot (\varrho \mathbf{v} \otimes \mathbf{v}) = \frac{\partial \varrho}{\partial t} \mathbf{v} + \varrho \cdot \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot (\varrho \mathbf{v}) \mathbf{v} + \varrho \mathbf{v} \cdot \nabla \mathbf{v} \quad (3)$$

With the continuity equation for the mass density $\partial \varrho / \partial t + \nabla \cdot (\varrho \mathbf{v}) = 0$ this term simplifies to

$$\varrho \cdot \frac{\partial \mathbf{v}}{\partial t} + \varrho \mathbf{v} \cdot \nabla \mathbf{v} \quad (4)$$

Thus we obtain the well-know form of the Navier-Stokes equations, or, preferring the singular form, the Navier-Stokes equation

$$\varrho \cdot \frac{\partial \mathbf{v}}{\partial t} + \varrho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p - \varrho \nabla \Phi + \nabla \cdot \mathbf{R} + \mathbf{F}_{\text{ext}} \quad (5)$$

where the term $-\varrho \nabla \Phi$ is gravity. If we neglect the viscosity term, we are left with the Euler equation

$$\varrho \cdot \frac{\partial \mathbf{v}}{\partial t} + \varrho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p - \varrho \nabla \Phi + \mathbf{F}_{\text{ext}} \quad (6)$$

Assumption 2

- We assume that $\mathbf{v}(\mathbf{r}, t)$ is independent of \mathbf{r} . In sharp distinction to the standard derivation of the barometric formulas, we relax the usual condition $\mathbf{v}(\mathbf{r}, t) \equiv 0$ in order to allow non-vanishing velocity fields $\mathbf{v}(\mathbf{r}, t)$, which are independent of \mathbf{r} .

Consequently, the viscosity tensor \mathbf{R} and the non-linear term $\varrho \mathbf{v} \cdot \nabla \mathbf{v}$ are zero, such that

$$\varrho \cdot \frac{\partial \mathbf{v}}{\partial t} = -\nabla p - \varrho \nabla \Phi + \mathbf{F}_{\text{ext}} \quad (7)$$

Remark: If one writes

$$\varrho \mathbf{v} \cdot \nabla \mathbf{v} = -\varrho \mathbf{v} \times (\nabla \times \mathbf{v}) + \varrho \nabla \left(\frac{1}{2} |\mathbf{v}|^2 \right) \quad (8)$$

one could weaken this assumption to potential velocity fields. With these formulas one can derive the Bernoulli equation.

In case of a rotating atmosphere of the Earth the last term \mathbf{F}_{ext} of the right hand side of Eq. 7 describes the centrifugal acceleration and the Coriolis acceleration. The latter vanishes for a identically vanishing velocity field.

Assumption 3

- We set \mathbf{F}_{ext} to zero.

We now have two fields, namely $-\nabla p$ and $-\rho \nabla \Phi$, which will accelerate the volume elements, if they are different fields:

$$\rho \cdot \frac{\partial \mathbf{v}}{\partial t} = -\nabla p - \rho \nabla \Phi \quad (9)$$

Let us follow the common notation and write for the gravitational field

$$\mathbf{g} = -\nabla \Phi \quad (10)$$

Assumption 4

- We assume that, as usual, acceleration due to gravity is vertical, i.e. we set

$$\mathbf{g} = -g \mathbf{e}_z \quad (11)$$

where \mathbf{e}_z is the unit vector in z -direction and g is constant in space and time. This is the flat earth hypothesis. Furthermore, we neglect the variation of the gravitation field induced by the gravitation fields of the Sun, the Moon, and the planets.

Assumption 5

- We assume that the wind blows only horizontally, i.e.

$$\mathbf{v} \cdot \mathbf{e}_z = 0 \quad (12)$$

Eq. 9 now becomes

$$\begin{pmatrix} \rho (\partial v_x / \partial t) \\ \rho (\partial v_y / \partial t) \\ 0 \end{pmatrix} = - \begin{pmatrix} \partial p / \partial x \\ \partial p / \partial y \\ \partial p / \partial z \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \rho g \end{pmatrix} \quad (13)$$

That is, with the usual assumptions about geometry we would get the hydrostatic equation

$$\frac{dp}{dz} = -\rho g \quad (14)$$

without the usual assumption $\mathbf{v}(\mathbf{r}, t) \equiv 0$. In standard thermodynamics for a macroscopic volume the pressure p is characterized by one number, not a field. Irreversible thermodynamics is a (classical) field theory and hydrodynamics is a special case.

2.2 Input from thermodynamics

The equation of state for an ideal gas reads

$$p v = \tilde{R} T \quad (15)$$

where v is the volume of one gram, and $\tilde{R} = R/(1 \text{ Mol})$, where R is the usual molar gas constant. Using the density ϱ we also may write, respectively,

$$\frac{p}{\varrho} = \tilde{R} T, \quad \varrho = \frac{p}{\tilde{R} T} \quad (16)$$

Assumption 6

- The air of the atmosphere obeys an equation of state of an ideal gas.

With $\varrho = p/(\tilde{R} T)$ inserted into Eq. 14 we have

$$\frac{dp}{dz} = -\frac{p g}{\tilde{R} T} = -\frac{(1 \text{ Mol}) p g}{\tilde{R} T} \quad (17)$$

If the molecular mass of the gas is greater, then the decrease of pressure with increase of height will be greater as well. For a temperature field T that is constant in space and time this equation can be integrated.

2.3 The isothermal atmosphere

Assumption 7a

- We postulate an isothermal atmosphere.

Separation of variable gives

$$\frac{d}{dz} \left(\ln \left(\frac{p}{p_0} \right) \right) = -\frac{g}{\tilde{R} T} \quad (18)$$

which may be integrated to

$$\left(\ln \left(\frac{p}{p_0} \right) \right) = -\frac{g}{\tilde{R} T} (z - z_0) \quad (19)$$

yielding

$$p = p_0 \exp \left(-\frac{g}{\tilde{R} T} (z - z_0) \right) \quad (20)$$

from which, with $\varrho = p/(\tilde{R} T)$, one obtains the density as a function of height

$$\varrho = \frac{p_0}{\tilde{R} T} \exp \left(-\frac{g}{\tilde{R} T} (z - z_0) \right) = \varrho_0 \exp \left(-\frac{g}{\tilde{R} T} (z - z_0) \right) \quad (21)$$

Thus, with help of these relations and assumptions, we obtain the barometric height formulas in case of an isothermal atmosphere. The lapse rates for the pressure and density, respectively, depend on the molecular mass of the gas, since $\tilde{R} = R/(1 \text{ Mol})$.

2.4 The adiabatic atmosphere

In what follows, we need three relations for the heat differential form dQ , namely

$$dQ = C_v(T) dT + \frac{\tilde{R}T}{v} dv \quad (22)$$

$$dQ = (C_v(T) + \tilde{R}) dT - \frac{\tilde{R}T}{p} dp \quad (23)$$

$$dQ = \frac{1}{\tilde{R}} \left(C_v \left(\frac{pv}{\tilde{R}} \right) + \tilde{R} \right) p dv + \frac{1}{\tilde{R}} C_v \left(\frac{pv}{\tilde{R}} \right) v dp \quad (24)$$

Assumption 7b

- For the ideal gas we use the reversible work form $p dv$.

We now calculate die adiabatic state changes, i.e. we set $dQ = 0$, and, separating the variables, we obtain

$$\int_{T_1}^{T_2} \frac{C_v(T)}{\tilde{R}T} dT = - \int_{v_1}^{v_2} \frac{dv}{v} = \ln \left(\frac{v_1}{v_2} \right), \quad (25)$$

$$\int_{T_1}^{T_2} \frac{C_v(T) + \tilde{R}}{\tilde{R}T} dT = \int_{p_1}^{p_2} \frac{dp}{p} = \ln \left(\frac{p_2}{p_1} \right) \quad (26)$$

Assumption 7c

- The specific heats of an ideal gas are independent of the absolute temperature.

Thus, we can continue our calculations with letters

$$\frac{C_v}{\tilde{R}} \int_{T_1}^{T_2} \frac{dT}{T} = \frac{C_v}{\tilde{R}} \ln \left(\frac{T_2}{T_1} \right) = \ln \left(\frac{v_1}{v_2} \right), \quad (27)$$

$$\frac{C_v + \tilde{R}}{\tilde{R}} \int_{T_1}^{T_2} \frac{dT}{T} = \frac{C_v + \tilde{R}}{\tilde{R}} \ln \left(\frac{T_2}{T_1} \right) = \ln \left(\frac{p_2}{p_1} \right), \quad (28)$$

For constant heat capacities we integrate the third equation (24) for $dQ = 0$. Since

$$0 = \frac{1}{\tilde{R}} (C_v + \tilde{R}) p dv + \frac{1}{\tilde{R}} C_v v dp \quad (29)$$

is equivalent to

$$0 = (C_v + \tilde{R}) p dv + C_v v dp \quad (30)$$

we get

$$(C_v + \tilde{R}) \int_{v_1}^{v_2} \frac{dv}{v} = -C_v \int_{p_1}^{p_2} \frac{dp}{p} \quad (31)$$

and, therefore,

$$\frac{C_v + \tilde{R}}{C_v} \ln \left(\frac{v_2}{v_1} \right) = \ln \left(\frac{p_1}{p_2} \right) \quad (32)$$

With our assumptions, we have

$$C_p - C_v = \tilde{R} \quad (33)$$

Setting

- $C_p/C_v = \kappa$,
- $T_2 = T$, $T_1 = T_0$,
- $p_2 = p$, $p_1 = p_0$,
- $v_2 = v$, $v_1 = v_0$,

we obtain the well-known *adiabatic equations of state*:

$$\kappa \cdot \ln \left(\frac{v}{v_0} \right) = \ln \left(\frac{p_0}{p} \right) \quad (34)$$

$$\left(\frac{v}{v_0} \right)^\kappa = \frac{p_0}{p} \quad (35)$$

$$\frac{p}{p_0} = \left(\frac{v}{v_0} \right)^{-\kappa} \quad (36)$$

and

$$\frac{C_v}{C_p - C_v} \cdot \ln \left(\frac{T}{T_0} \right) = \ln \left(\frac{v_0}{v} \right) \quad (37)$$

$$\frac{1}{\kappa - 1} \ln \left(\frac{T}{T_0} \right) = \ln \left(\frac{v_0}{v} \right) \quad (38)$$

$$\left(\frac{T}{T_0} \right)^{1/(\kappa-1)} = \frac{v_0}{v} \quad (39)$$

and

$$\frac{C_v + \tilde{R}}{\tilde{R}} \cdot \ln \left(\frac{T}{T_0} \right) = \ln \left(\frac{p}{p_0} \right) \quad (40)$$

$$\frac{\kappa}{\kappa - 1} \cdot \ln \left(\frac{T}{T_0} \right) = \ln \left(\frac{p}{p_0} \right) \quad (41)$$

$$\frac{p}{p_0} = \left(\frac{T}{T_0} \right)^{\kappa/(\kappa-1)} \quad (42)$$

These adiabatic equations of state are well-known from standard textbooks. However, almost never the assumptions are discussed, under which they hold true.

Now we replace the one-gram volume v by the density ρ . We get

$$\frac{p}{p_0} = \left(\frac{\varrho}{\varrho_0} \right)^\kappa \quad (43)$$

$$\frac{\varrho}{\varrho_0} = \left(\frac{T}{T_0} \right)^{1/(\kappa-1)} \quad (44)$$

$$\frac{p}{p_0} = \left(\frac{T}{T_0} \right)^{\kappa/(\kappa-1)} \quad (45)$$

Rewriting the first of these three equations as

$$p = \left(\frac{p_0}{\varrho_0^\kappa} \right) \varrho^\kappa \quad (46)$$

and inserting it into Eq. 14, namely $dp/dz = -\varrho g$, whereby we consider p and ϱ as functions of z , we obtain, applying the chain rule of differential calculus,

$$\frac{dp}{dz} = \kappa \frac{p_0}{\varrho_0^\kappa} \varrho^{\kappa-1} \frac{d\varrho}{dz} = -\varrho g \quad (47)$$

$$\kappa \frac{p_0}{\varrho_0^\kappa} \varrho^{\kappa-2} \frac{d\varrho}{dz} = -g \quad (48)$$

$$\frac{d}{dz}(\varrho^{\kappa-1}) = -\frac{(\kappa-1)g\varrho_0^\kappa}{\kappa_0 p_0} \quad (49)$$

Integrating this equation we obtain the density ϱ as a function of height

$$\varrho^{\kappa-1} = \varrho_0^{\kappa-1} - \frac{(\kappa-1)g\varrho_0^\kappa}{\kappa p_0} (z - z_0) \quad (50)$$

$$\varrho(z) = \varrho_0 \left(1 - \frac{(\kappa-1)\varrho_0 g}{\kappa p_0} (z - z_0) \right)^{1/(\kappa-1)} \quad (51)$$

From this and Eq. 46 we get the pressure p as a function of height z

$$p(z) = p_0 \left(1 - \frac{(\kappa-1)\varrho_0 g}{\kappa p_0} (z - z_0) \right)^{\kappa/(\kappa-1)} \quad (52)$$

Inserting this into $p/p_0 = (T/T_0)^{\kappa/(\kappa-1)}$ resp. $T = T_0(p/p_0)^{(\kappa-1)/\kappa}$ we obtain the temperature T as a function of the height z

$$T(z) = T_0 \left(1 - \frac{(\kappa-1)\varrho_0 g}{\kappa p_0} (z - z_0) \right) = T_0 - \frac{(\kappa-1)g}{\kappa \tilde{R}} (z - z_0) \quad (53)$$

We conclude:

- The temperature is decreasing linearly with increasing height.

As in the case of an isothermal atmosphere the lapse rate will be higher if the molecular mass is greater. In the adiabatic case this slope depends on the adiabatic coefficient κ . Note, that one can always weaken the assumption to get similar results (e.g. moist adiabatic approximation and so on).

With our assumptions we may rewrite the constants appearing in the formula before the height term, i.e.

$$\frac{(\kappa - 1)\varrho_0 g}{\kappa p_0} = \frac{(C_p/C_v - 1)\varrho_0 g}{(C_p/C_v)p_0} \quad (54)$$

$$= \frac{(C_p - C_v)\varrho_0 g}{C_p p_0} \quad (55)$$

$$= \frac{(C_p - C_v)g}{C_p \tilde{R} T_0} \quad (56)$$

$$= \frac{g}{C_p T_0} \quad (57)$$

and

$$\frac{(\kappa - 1)g}{\kappa \tilde{R}} = \frac{(C_p - C_v)g}{C_p \tilde{R}} = \frac{g}{C_p} \quad (58)$$

Finally, with these coefficients the adiabatic height formulas read

$$\varrho(z) = \varrho_0 \left(1 - \frac{g}{C_p T_0}(z - z_0)\right)^{C_v/(C_p - C_v)} \quad (59)$$

$$p(z) = p_0 \left(1 - \frac{g}{C_p T_0}(z - z_0)\right)^{C_p/(C_p - C_v)} \quad (60)$$

$$T(z) = T_0 - \frac{g}{C_p}(z - z_0) \quad (61)$$

3 Results

By combining hydrodynamics, thermodynamics, and imposing the above listed assumptions for planetary atmospheres one can compute the temperature profiles of idealized atmospheres. In case of the adiabatic atmosphere the decrease of the temperature with height is described by a linear function with slope $-g/C_p$, where C_p depends weakly on the molecular mass. As elucidated in our paper [4, 3] mixtures of gases are covered in the context of Gibbs thermodynamics. Since the measurable thermodynamic quantities of a voluminous medium, in particular the specific heat and the thermodynamic transport coefficients, naturally include the contribution from radiative interactions, we cannot expect that a change of concentration of a trace gas has any measurable effect. At this point, it is important to remember that the barometric formulas do not hold globally but have only a limited range of validity.

Appendix: Relevance to the current climate debate

In our falsification paper [4, 3] we have shown that the atmospheric CO₂ greenhouse effects [5] as taken-for-granted concepts in global climatology do not fit into the scientific framework of theoretical and applied physics. By showing that

- (a) there are no common physical laws between the warming phenomenon in glass houses and the fictitious atmospheric greenhouse effects
- (b) there are no calculations to determine an average surface temperature of a planet
- (c) the frequently mentioned difference of 33 degrees Celsius is a meaningless number calculated wrongly
- (d) the formulas of cavity radiation are used inappropriately
- (f) the assumption of a radiative balance is unphysical
- (e) thermal conductivity and friction must not be set to zero

the atmospheric CO₂ greenhouse effects have been refuted within the frame of physics [4, 3].

In other words, the greenhouse models are all based on simplistic pictures of radiative transfer and their obscure relation to thermodynamics, disregarding the other forms of heat transfer such as thermal conductivity, convection, latent heat exchange *et cetera*. Some of these simplistic descriptions define a “Perpetuum Mobile Of The 2nd Kind” and are therefore inadmissible as a physical concept.

In the speculative discussion around the existence of an atmospheric natural greenhouse effect [6] or the existence of an atmospheric CO₂ greenhouse effect it is sometimes stated that the greenhouse effect could modify the temperature profile of the Earth’s atmosphere. This conjecture is related to another popular but incorrect idea communicated by some proponents of the global warming hypothesis, namely the hypothesis that the temperatures of the Venus are due to a greenhouse effect. For instance, in their book “Der Klimawandel. Diagnose, Prognose, Therapie” (Climate Change. Diagnosis, Prognosis, Therapy) “two leading international experts”, Hans-Joachim Schellhuber and Stefan Rahmstorf, present a “compact and understandable review” of “climate change” to the general public [8]. On page 32 they explicitly refer to the “power” of the “greenhouse effect” on the Venus.

The claim of Rahmstorf and Schellhuber is that the high venusian surface temperatures somewhere between 400 and 500 Celsius degrees are due to an atmospheric CO₂ greenhouse effect [8]. Of course, they are not. On the one hand, since the venusian atmosphere is

opaque to visible light, the central assumption of the greenhouse hypotheses is not obeyed. On the other hand, if one compares the temperature and pressure profiles of Venus and Earth, one immediately will see that they are both very similar. An important difference is the atmospheric pressure on the ground, which is approximately two orders higher than on the Earth. At 50 km altitude the venusian atmospheric pressure corresponds to the normal pressure on the Earth with temperatures at approximately 37 Celsius degrees. However, things are extremely complex (volcanic activities, clouds of sulfuric acid), such that we do not go in details here [9].

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