The Christiansen relation for radiation between two plates with different surface conditions, including the derivation of the Kirchhoff relation

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We will use the symbols

- ε for emission coefficient, representing the fraction of emitted energy for non-black surfaces compared to the emitted energy of black surfaces. For black surfaces $\varepsilon = 1$ and the energy emitted by a black surface is $\theta = \sigma T^4$. (Stefan-Boltzmann relation).
- α for absorption coefficient, representing the fraction of energy absorbed as compared to total energy arriving at the surface.
- ρ for reflection coefficient, representing the fraction of energy reflected as compared to total energy arriving at the surface.
- τ for transmission coefficient, representing the fraction of energy transmitted through the body behind a surface as compared to the energy arriving at the surface.

From the definitions of α, τ, ρ it follows:

$$\alpha + \tau + \rho = 1 \; . \qquad \textbf{(1)}$$

Consider surfaces 1 and 2, with temperatures T_1 and T_2, black-surface radiations θ_1 and θ_2, Emission coefficients (ε_1, ε_2), Absorption coefficients (α_1, α_2), Reflection coefficients (ρ_1, ρ_2) and Transmission coefficients (τ_1, τ_2).

We follow in **Table 1** the history of an emission:

$$EM_1 = \varepsilon_1 \theta_1 \qquad \textbf{(2)}$$

from surface 1 in the direction of surface 2.

Table 1

			reflection from 1 or 2	absorption in 1	transmitted through 2	transmitted through 1
1	Fraction ρ_2 of	$\varepsilon_1 \theta_1$				
	is reflected from 2 to 1		$\rho_2 \varepsilon_1 \theta_1$			
2	Fraction τ_2 of	$\varepsilon_1 \theta_1$				
	is transmitted through 2				$\tau_2 \varepsilon_1 \theta_1$	
3	Fraction α_1 of	$\rho_2 \varepsilon_1 \theta_1$				
	is absorbed at surface 1 :			$\alpha_1 \rho_2 \varepsilon_1 \theta_1$		
4	Fraction τ_1 of	$\rho_2 \varepsilon_1 \theta_1$				
	is transmitted through 1 :					$\tau_1 \rho_2 \varepsilon_1 \theta_1$
5	Fraction ρ_1 of	$\rho_2 \varepsilon_1 \theta_1$				
	is reflected from 1 to 2 :		$\rho_1 \rho_2 \varepsilon_1 \theta_1$			

Table 1 continued

Steps 1, 2, 3, 4 and 5 are repeated several times:

		reflection from 1 or 2	absorption in 1	transmitted through 2	transmitted through 1
1bis	Fraction ρ_2 of $\rho_1\rho_2\,\varepsilon_1\theta_1$ is reflected from 2 to 1	$\rho_2\rho_1\rho_2\,\varepsilon_1\theta_1$			
2bis	Fraction τ_2 of $\rho_1\rho_2\,\varepsilon_1\theta_1$ is transmitted through 2			$\tau_2\,\rho_1\rho_2\,\varepsilon_1\theta_1$	
3bis	Fraction α_1 of $\rho_2\rho_1\rho_2\,\varepsilon_1\theta_1$ is absorbed at surface 1 :		$\alpha_1\rho_2\rho_1\rho_2\,\varepsilon_1\theta_1$		
4bis	Fraction τ_1 of $\rho_2\rho_1\rho_2\,\varepsilon_1\theta_1$ is transmitted through 1 :				$\tau_1\rho_2\rho_1\rho_2\,\varepsilon_1\theta_1$
5bis	Fraction ρ_1 of $\rho_2\rho_1\rho_2\,\varepsilon_1\theta_1$ is reflected from 1 to 2 :	$\rho_1\rho_2\rho_1\rho_2\,\varepsilon_1\theta_1$			

etc , etc

The total absorption in 1 becomes : $\alpha_1\rho_2\,\varepsilon_1\,\theta_1\,(1+\ \rho_1\rho_2 + (\ \rho_1\rho_2\,)\verb|^|2 +\ \ (\ \rho_1\rho_2)\verb|^|3 + \ldots)$

The terms in parentheses represent geometrical series with ratio $\rho_1\rho_2 < 1$ with sum: $1/(1-\ \rho_1\rho_2)$

Total Absorption in 1 :

$$AB_1 = \ \alpha_1\rho_2\,\varepsilon_1\,\theta_1\,/(1-\ \rho_1\rho_2) \qquad \textbf{(3)}$$

The transmission through 1 becomes: $\tau_1\rho_2\,\varepsilon_1\theta_1\,(1+\ \rho_1\rho_2 + (\ \rho_1\rho_2\,)\verb|^|2 +\ \ (\ \rho_1\rho_2)\verb|^|3 + \ldots)$

Total transmission through 1:

$$TR_1 = \ \tau_1\rho_2\,\varepsilon_1\theta_1\,/(1-\ \rho_1\rho_2) \qquad \textbf{(4)}$$

The transmission through 2 becomes: $\tau_2\,\varepsilon_1\theta_1\,(1+\ \rho_1\rho_2 + (\ \rho_1\rho_2\,)\verb|^|2 +\ \ (\ \rho_1\rho_2)\verb|^|3 + \ldots)$

Total transmission through 2:

$$TR_2 = \tau_2\,\varepsilon_1\theta_1/(1-\ \rho_1\rho_2) \qquad \textbf{(5)}$$

The energy flux q1 from 1 to 2 becomes from Emission minus Total Absorption :EM_1-AB_1

$$q_1 = \ \varepsilon_1\theta_1(1-\alpha_1\rho_2\,/(1-\rho_1\rho_2)) \qquad \textbf{(6)}$$

By cyclic permutation of the subscripts, the energy flux q_2 from 2 to 1:

$$q_2 \ = \varepsilon_2\,\theta_2(1-\alpha_2\rho_1\,/(1-\rho_1\rho_2)) \qquad \textbf{(7)}$$

The relations (6) and (7) for q_1 respectively q_2 are valid for arbitrary values of α, τ , ρ within the interval 0 to 1, satisfying the relation (1): $\alpha+\tau+\rho=1$. The relations do not contain the transmission coefficients τ explicitly, they are however included through the reflection coefficients ρ !

Attention: These relations are only valid for black or gray surfaces, not for selective radiadiation. For selective radiators the equation have to be established for each wavelength interval.

Case of two non-transparent plates with $\tau 1 = \tau 2 = 0$

In this case according to relation **(1)** $\rho_1 = 1 - \alpha_1$ and $\rho 2 = 1 - \alpha_2$.
Inserting these values in **(6)** respectively **(7)**:

$$q_1 = \varepsilon_1 \alpha_2 \theta_1 / (\alpha_1 + \alpha_2 - \alpha_1 \alpha_2) \quad \textbf{(8)}$$

$$q_2 = \varepsilon_2 \alpha_1 \theta_2 / (\alpha_1 + \alpha_2 - \alpha_1 \alpha_2) \quad \textbf{(9)}$$

In case, 1 is black ($\varepsilon_1 = \alpha_1 = 1$) and 2 is gray and $\theta_1 = \theta_2 = \theta$: $q_1 = \alpha_2 \theta$ and $q_2 = \varepsilon_2 \theta$.
The temperatures of the surfaces are equal and material behind the surfaces are also isotherm: consequently no heat flow between 1 and 2: $q_1 - q_2 = 0$.
From which follows for a gray surface it turns out that $\alpha_2 = \varepsilon_2$.
Since q_1 is independent of θ_2 it is also true for $\theta_1 \neq \theta_2$, from which follows for gray non transparent surfaces with $\tau = 0$, and even where α and ε are not equal to 1:

$$\alpha = \varepsilon \quad \textbf{(10)}$$

This is the Kirchhoff relation of 1860!

The net heat flux between surface 1 and 2 becomes :

$$q_1 - q_2 = (\alpha_1 \alpha_2 / (\alpha_1 + \alpha_2 - \alpha_1 \alpha_2)) (\theta_1 - \theta_2) \quad \textbf{(11)}$$

With $\sigma_1 = \alpha_1 \sigma$, $\sigma_2 = \alpha_2 \sigma$ and $1/\sigma_{12} = 1/\sigma_1 + 1/\sigma_2 - 1/\sigma$

$$q_1 - q_2 = \sigma_{12}(T_1^4 - T_2^4) \quad \textbf{(12)}$$

This is the Christiansen relation
C Christiansen, Annalen der Physik und Chemie, Leipzig,1883

Application of the Christiansen relation see:
http://www.tech-know-group.com/papers/Prevost_no_back-radiation-v2.pdf

Case of a non-transparent plate ($\tau_1 = 0$) and a semi-transparent plate ($\tau_2 > 0$).

In this case we get from equation **(1)**: $\rho_1 = 1 - \alpha_1$ and $\rho_2 = 1 - \alpha_2 - \tau_2$
Inserting these value in **(6)** respectively **(7)** we get for the energy fluxes:

$$q_1 = \varepsilon_1 \theta_1 (\alpha_2 + \tau_2) / ((\alpha_1 + \alpha_2 - \alpha_1 \alpha_2) + \tau_2 (1 - \alpha_1)) \quad \textbf{(13)}$$

$$q_2 = \varepsilon_2 \theta_2 (\alpha_1 + \tau_2 (1 - \alpha_1)) / ((\alpha_1 + \alpha_2 - \alpha_1 \alpha_2) + \tau_2 (1 - \alpha_1)) \quad \textbf{(14)}$$

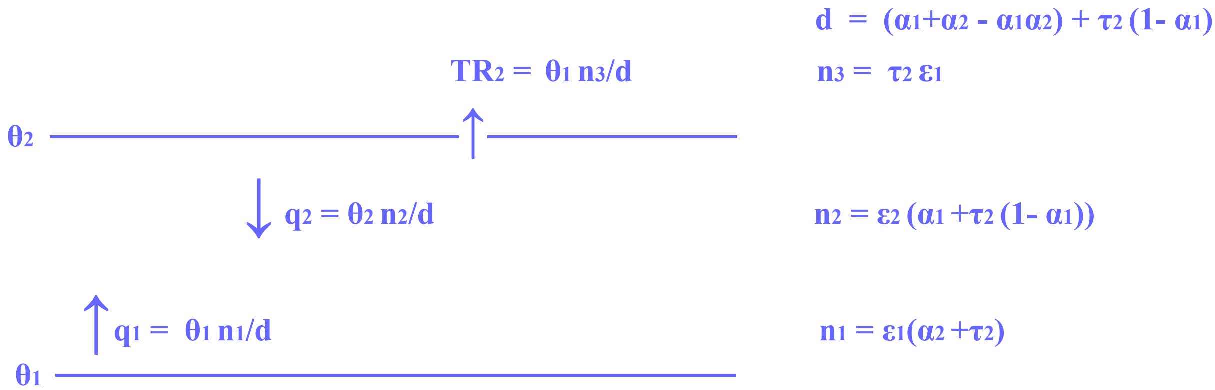
With numerator 1 = $n_1 = \varepsilon_1 (\alpha_2 + \tau_2)$
numerator 2 = $n_2 = \varepsilon_2 (\alpha_1 + \tau_2 (1 - \alpha_1))$
numerator 3 = $n_3 = \tau_2 \varepsilon_1$
denominator = $d = (\alpha_1 + \alpha_2 - \alpha_1 \alpha_2) + \tau_2 (1 - \alpha_1)$

$$q_1 = \theta_1 n_1 / d \quad \textbf{(13a)}$$

$$q_2 = \theta_2 n_2 / d \quad \textbf{(14a)}$$

$$TR_2 = \theta_1 n_3 / d \quad \textbf{(5a)}$$

Figure 1 Various energy fluxes for an emission $EM_1 = \varepsilon_1\theta_1$, $\tau_1 = 0$ **and** $\tau_2 > 0$



$q_{net}(1\text{into } 2) = q_1 - TR_2 - q_2$

In **Figure 1** are depicted the various energy fluxes as function of the emission coefficients ε_1 and ε_2, the absorption coefficients α_1 and α_2 and the transmission coefficient τ_2 .
For $\tau_2 = 0$ the transmitted energy trough plate 2 $TR_2 = 0$ and q1 and q2 from equations (13) and (15) are identical to q_1 and q_2 from equations (8) and (9).
The latter equations gave the Kirchhoff relation $\varepsilon_2 = \alpha_2$.
In case $\tau_2 > 0$, a Kirchhoff type of relation relation follows from $q_1 - TR_2 = q_2$,
We compare the numerators of the terms and find a relation between ε_1/α_1 and ε_2/α_2 :

$$n_1 - n_2 = n_3 \quad \text{or} \quad \varepsilon_2/\alpha_2 = \varepsilon_1/\alpha_1\,(\alpha_1/(\alpha_1 + \tau_2(1-\alpha_1)) \qquad \textbf{(15)}$$

This is the modification of the Kirchhoff relation for the case that surface 1 is non-transparent and surface 2 is semi-transparent with transmission coefficients $\tau_1 = 0$ respectively $\tau_2 > 0$.
In case surface 1 is black we find $\varepsilon_1 = \alpha_1 = 1$ and $\varepsilon_2/\alpha_2 = 1$.
In case surface 2 is non-transparent ($\tau_2 = 0$) we find $\varepsilon_2/\alpha_2 = \varepsilon_1/\alpha_1 = 1$.
Introducing (15) into the equations 13 and 14 we find for qnet the non-transparent plate 1 and the semi-tranparent plate 2:

$$q_{net}(1\text{into } 2) = \varepsilon_1\alpha_2/(\alpha_1+\alpha_2 - \alpha_1\alpha_2 + \tau_2(1-\alpha_1))\,(\theta_1 - \theta_2) \qquad \textbf{(16a)}$$

or equivalently due to the modified Kirchhoff relation **(15)**

$$q_{net}(1\text{into } 2) = \varepsilon_2(\alpha_1+\tau_2(1-\alpha_1))/(\alpha_1+\alpha_2 - \alpha_1\alpha_2 + \tau_2(1-\alpha_1))\,(\theta_1 - \theta_2) \qquad \textbf{(16b)}$$

For $\tau_2 = 0$ both equations reduce to the Christansen relation **(12)**:

$$q_{net}(1 \text{ into } 2) = q_1 - q_2 = \alpha_{12}(\theta_1 - \theta_2) \quad \text{with} \quad 1/\alpha_{12} = 1/\alpha_1 + 1/\alpha_2 - 1 \qquad \textbf{(12bis)}$$

or equivalently

$$q_1 - q_2 = \sigma_{12}(T_1\text{^}4 - T_2\text{^}4) \quad \text{with} \quad \sigma_1 = \alpha_1\sigma,\ \sigma_2 = \alpha_2\sigma \ \text{ and } 1/\sigma_{12} = 1/\sigma_1 + 1/\sigma_2 - 1/\sigma \qquad \textbf{(12bis)}$$

Attention: Above equations are valid for black and gray surfaces.
For selective radiators the equations have to be established for each wavelength interval.